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Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel

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A blob of Newtonian fluid is sandwiched in the narrow gap between two plane parallel surfaces so that, at some initial instant, its plan-view occupies a simply connected domain D_0. Further fluid, with the same material properties, is injected into the gap at some fixed point within D_0, so that the blob begins to grow in size. The domain D occupied by the fluid at some subsequent time is to be determined.

It is shown that the growth is controlled by the existence of an infinite number of invariants of the motion, which are of a purely geometric character. For sufficiently simple initial domains D_0 these allow the problem to be reduced to the solution of a finite system of algebraic equations. For more complex initial domains an approximation scheme leads to a similar system of equations to be solved.

1. Introduction

One of the basic manufacturing processes used in the plastics industry is that of injection moulding. Molten polymer is forced into a mould of an appropriate shape through a strategically placed hole, and subsequently allowed to solidify. As a simple example, one might consider the production of a plane lamina: the hollow of the mould would consist of the narrow gap between two parallel planes, with side-wall boundaries enclosing a void whose plan-form coincided with that of the required lamina. In order to reduce the high pressures needed to force the melt into the mould as much as possible, the injection point would normally be in one of the plane faces, somewhere near its centre. In the early stages of filling, symmetry considerations suggest that the plan-view of the region occupied by melt will be an expanding circle, this situation continuing until the melt reaches the nearest side wall and begins to move along it. An analysis of the subsequent motion would obviously be complex but, nevertheless, highly desirable. For example, the mould can only be filled successfully if, at the same time, we allow the air to escape, and this must be done through air vents placed at those points which will be the last to be filled. How can these points be determined, *a priori*, for a given mould?

In the above industrial situation a large number of complex physical phenomena are at work: the high pressures involved imply that compressibility effects may be important; the high viscosity of the melt will give rise to significant heat generation and associated thermal effects; and the polymer melt will behave as a non-Newtonian fluid whose relevant rheology we know little about. One would, therefore, first of all, wish to consider the same basic problem with these complications removed. We replace the polymer melt by an incompressible Newtonian fluid, and assume thermal effects to be negligible. With a mould appropriate for a plane lamina, the problem then becomes one of Hele Shaw flow with free boundaries, and the simple nature of the equations involved encourages one to hope for a satisfactory analytical solution.

As a preliminary to the above, we here consider the situation when the mould is devoid of side walls: the fluid expands into a narrow gap which is of infinite extent. Starting with an initially empty channel, one would anticipate the trivial solution in which the blob of fluid has a plan-view in the form of an expanding circle. We, therefore, consider instead that, at some initial instant, the plan occupies some given simply connected domain, and we wish to determine the way this domain expands as further fluid is injected at some fixed point in its interior. Although the movement of the free boundary is governed by nonlinear conditions, we are able to show that the motion has an infinite number of invariants. Exploiting these, the problem can be reduced to the solution of a finite system of algebraic equations, either exactly, for certain simple initial domains, or approximately, in general. Thus, in spite of the inherent nonlinearities, one can successfully predict the shape of the blob at subsequent times, given the initial shape.

Hele Shaw flows with a free boundary have been considered by Saffman & Taylor (1958), Taylor & Saffman (1959), Taylor (1961) and Jacquard & Séguier (1962). These authors were primarily concerned with problems arising from flow in a porous medium, but exploited the fact that the two physical situations are mathematically analogous. The present results may also be interpreted in terms of free-boundary flows in a porous medium.

It is perhaps worth pointing out here that the usual assumptions made to analyse two-dimensional problems of electrochemical machining (Krylov 1968; Fitz-Gerald, McGeough & Marsh 1969; Fitz-Gerald & McGeough 1969, 1970; Collett, Hewson-Browne & Windle 1970) also lead to mathematical problems identical with those arising from certain Hele Shaw flows with free boundaries. The flows considered in the present paper, corresponding to the use of a long, thin, cylindrical cathode, have little relevance in the machining context, but the analogy could prove useful experimentally in the design of such machine tools.

2. Formulation

We consider motion in the narrow gap between two parallel planes a distance h apart. Take Cartesian co-ordinates $(x, y, \tilde{z})$, the $\tilde{z}$ axis being perpendicular to the planes, so that the plan-view of the blob of Newtonian fluid sandwiched between them is projected onto the z plane, where $z = x+iy$. We assume this projection to be simply connected and to occupy a domain D of the z plane. We take the origin to be the injection point, so that $z = 0$ is an interior point of D. D_0 will refer to the domain occupied by the blob in the initial state.

Away from the injection point and free boundary, we can use the standard arguments from the theory of the Hele Shaw cell (see Lamb 1932, p. 582 for example). Ignoring gravitational effects, a narrow-gap assumption leads to the conclusion that the pressure is independent of $\tilde{z}$, and that the velocity in the fluid is everywhere in the direction of the pressure gradient, varying in a parabolic manner between the planes. This allows the velocity to be averaged over the depth to remove the dependence on $\tilde{z}$. The x and y components of this averaged velocity, u and v, are given by

$$\mathbf{u} = (u, v) = -(h^2/12\mu)\nabla p, \tag{2.1}$$

where μ is the viscosity and p the pressure. This is just $\mathbf{u} = \nabla\phi$, where ϕ is a constant multiple of the pressure, so that the pressure essentially provides a velocity potential. Incompressibility implies the existence of a stream function $\psi(x, y)$ for the averaged velocity, so that

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}; \quad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{2.2}$$

These are the Cauchy–Riemann equations, implying that $w(z) = \phi + i\psi$ is an analytic function of z. In terms of this complex potential $w(z)$ we have

$$u - iv = w'(z). \tag{2.3}$$

This analysis can be expected to hold only up to distances of order h from the injection point and the free boundary. The details near the injection point at the origin will depend on the precise mode of filling but, away from its immediate vicinity, we can model it as a point source in the flow by requiring

$$w(z) \sim (Q/2\pi)\log z \quad \text{as} \quad |z| \to 0, \tag{2.4}$$

where Q is the rate of area increase of the blob, this being equal to q/h, where q is the volume input rate.

The situation near the advancing free boundary is complex. First, because of meniscus effects, there will be an ambiguity in its position of order h. However, for a small gap, such an error will be immaterial. Second, the parabolic velocity distribution across the gap must be modified near the boundary. Nevertheless, continuity considerations show that the velocity of advance of the boundary can be equated to the average velocity $\mathbf{u}$ across the gap in that neighbourhood, with an expected error of order h.

A third problem concerns the stress condition to be applied at the free boundary. If surface tension is neglected, one would expect the pressure just inside the fluid to equal that just outside, with an error of order h. Surface tension will give rise to a pressure drop across this interface which is dependent on its principal radii of curvature. However, one of these can be expected to be of order h, while the other will be the radius of curvature of the boundary of the domain D. Provided that the latter is much larger than h at every point, the pressure drop will be essentially independent of the particular point on the boundary. We would, therefore, still expect a constant pressure condition to be appropriate. This aspect is fully discussed by Saffman & Taylor (1958), Taylor & Saffman (1959), and Taylor (1961) whose experiments confirm this assumption, provided the velocities

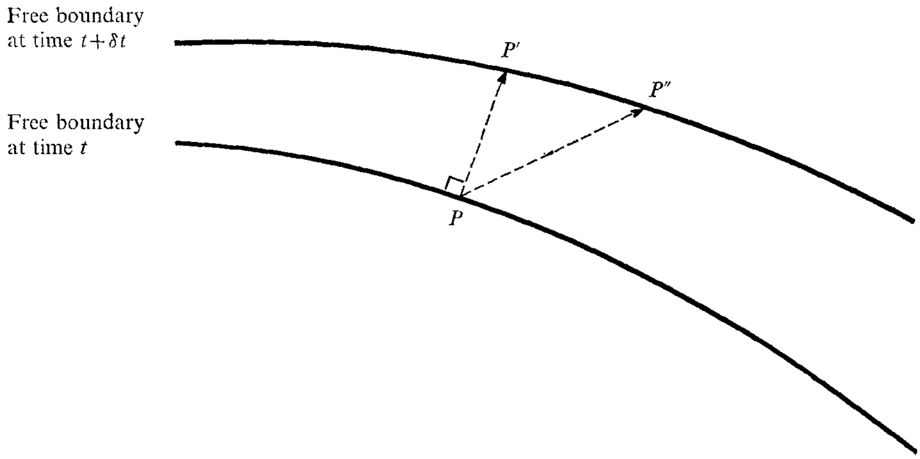

FIGURE 1. Sketch establishing notation for the derivation of the kinematic condition at the free boundary.

involved are not too small. We therefore impose the condition that the pressure, and hence the velocity potential ϕ, should be a constant on the free surface, and we may take this constant to be zero.

Employing this last assumption, if we know the domain D occupied by the blob at some time, we can then determine the flow pattern at that time. Let the domain D in the z plane be mapped conformally onto the interior of the unit circle in the ζ plane by means of $z = f(\zeta)$, where $f(\zeta)$ is analytic within the unit circle and $f(0) = 0$, so that the origins correspond. Defining $\Omega(\zeta) = w\{f(\zeta)\}$, the transformed problem in the ζ plane has the solution

$$\Omega(\zeta) = (Q/2\pi)\log\zeta, \tag{2.5}$$

thus determining $w(z)$.

We now recognize that D, and therefore $f(\zeta) = f(\zeta, t)$, actually depend on time. The velocity at the boundary, determined by the above solution, gives the rate at which that boundary is advancing. To determine the form taken by this kinematic condition we refer to figure 1. Point P, at position z on the free boundary at time t, corresponds to the point ζ on the unit circle under $z = f(\zeta, t)$. The fluid velocity at P is perpendicular to the free surface, so that, at time $t+\delta t$, the point P has moved to P', where

$$PP' = \overline{w'(z)}\,\delta t + O(\delta t^2),$$

an overbar being used to denote the complex conjugate. However, at time $t+\delta t$, the same point ζ on the unit circle corresponds to the point P'', where

$$PP'' = (\partial f/\partial t)\,\delta t + O(\delta t^2).$$

The projection of PP'' on the direction of PP' must be the length of PP' to $O(\delta t)$, so that $\operatorname{Re}\{w'(z)\,\partial f/\partial t\} = |w'(z)|^2$ on the free boundary. Transforming from the z plane to the ζ plane and using (2.5), this boundary condition becomes

$$f'(\zeta,t)\frac{\partial \bar{f}}{\partial t} + \overline{\zeta^2 f'(\zeta,t)}\frac{\partial f}{\partial t} = \frac{Q}{\pi}\frac{1}{\zeta} \quad \text{on} \quad |\zeta| = 1, \tag{2.6}$$

where the primes here denote derivatives with respect to ζ.

The dependence of the mapping function on time is governed entirely by condition (2.6). In spite of its apparent complexity, the manner by which it controls the growth of the domain D can be expressed in a very simple form.

3. The moment constants

With the domain D we associate an infinite sequence of moment constants C_N; $N = 0, 1, 2, \ldots$, where the Nth moment constant is defined by

$$C_N = \iint_D z^N \, dx \, dy. \tag{3.1}$$

Thus C_0 is simply the area of D, while C_1/C_0 gives the position of its centre of area.

Using Γ to denote the boundary of D and Γ' to denote the unit circle in the ζ plane, both traversed in an anticlockwise direction, we can use Green's theorem to write C_N as a line integral round either of these boundaries. In fact

$$C_N = \frac{1}{2i}\int_\Gamma z^N \bar{z} \, dz = \frac{1}{2i}\int_{\Gamma'} f^N \bar{f} f' \, d\zeta. \tag{3.2}$$

As the domain D expands, we must expect these moment constants to depend on time. Differentiation of (3.2) gives

$$2i\frac{dC_N}{dt} = \int_{\Gamma'} N f^{N-1} f' \bar{f} \frac{\partial f}{\partial t} d\zeta + \int_{\Gamma'} f^N f' \frac{\partial \bar{f}}{\partial t} d\zeta + \int_{\Gamma'} f^N \bar{f} \frac{\partial f'}{\partial t} d\zeta.$$

In the first term on the right, we integrate by parts, integrating the combination $Nf^{N-1}f'$. One of the resulting terms cancels with the third term on the right while for the other we note that $\partial/\partial\zeta \equiv -\bar{\zeta}^2 \, \partial/\partial\bar{\zeta}$ when the differentiation is carried out round the unit circle $\zeta\bar{\zeta} = 1$. There remains

$$\begin{aligned}\frac{dC_N}{dt} &= \frac{1}{2i}\int_{\Gamma'} f^N \left\{ f' \frac{\partial \bar{f}}{\partial t} + \overline{\zeta^2 f'} \frac{\partial f}{\partial t} \right\} d\zeta \\ &= \frac{Q}{2\pi i}\int_{\Gamma'} \frac{f^N}{\zeta} d\zeta \quad \text{using (2.6).}\end{aligned}$$

But f is analytic within Γ' and has a simple zero at the origin, so that the right-hand side here takes the value Q for $N = 0$ and vanishes for $N = 1, 2, \ldots$. That C_0, being the area of D, grows at the rate Q is to be expected: the above shows that the remaining C_N are invariants. The growth is thus controlled by the restrictions

$$C_N = \text{constant in time for} \quad N = 1, 2, 3, \ldots. \tag{3.3}$$

Given the initial domain, we can evaluate the initial values of the moment constants. At a later time, when the area of the blob has been increased by A, the value of C_0 will have been increased by A, but the remainder will retain their initial values. One thus needs to be able to construct a domain D, given its sequence of moment constants.

In this form, the problem is reminiscent of the classical moment problems of Stieltjes, Hamburger and Hausdorff (see Widder 1946; Akhiezer 1965, for example) and one might pose similar purely mathematical questions. What conditions on a sequence of complex numbers are necessary and sufficient to ensure that a simply connected single-sheeted domain D exists for which they are the moment constants? When such a domain exists, is it unique? How does one construct the domain from the sequence? In the present context, however, these questions need not concern us, for we know the initial domain D_0: we require to determine the manner in which continuous changes occur when C_0 is increased, the other moment constants being kept fixed.

4. Reduction to a functional equation

We first suppose that the domain is bounded, and enclosed within a circle of radius R centred at the origin of the z plane. In fact, this assumption is almost essential for the moment constants to be well-defined. Unbounded domains may be treated as limiting cases of bounded domains if desired, as in the example of § 5.

The definition of C_N by (3.1) gives the estimate

$$|C_N| \leqslant R^N C_0, \tag{4.1}$$

which allows us to assert that the infinite series

$$h(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{C_n}{z^{n+1}} \tag{4.2}$$

converges for $|z| > R$, and therefore represents, in this region, an analytic function of z which vanishes at infinity.

Using the first equality of (3.2) we can, for $|z| > R$, write (4.2) as

$$h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\bar{\tau}}{z-\tau} d\tau. \tag{4.3}$$

It now follows that $h(z)$ can, in fact, be continued analytically into the whole of the region exterior to D: the function $h(z)$, initially defined by (4.2) in terms of the moment constants, has all its singularities within D.

We now make the assumption that Γ is an analytic curve, i.e. can be written as

$$\Gamma: \quad \bar{z} = g(z), \tag{4.4}$$

where $g(z)$ is analytic in a neighbourhood of the curve. Even if we were to pose initial conditions with an initial Γ_0 which is not analytic, it is conjectured that the curve at any later time will, in fact, be analytic. A non-analytic initial Γ_0 can be dealt with by regarding it as the limit of a sequence of analytic curves. This assumption is equivalent to requiring that the mapping $z = f(\zeta)$ from D to the unit disk be such that $f(\zeta)$ is analytic in a neighbourhood of every point of the unit circle.

We may write

$$g(z) = g_i(z) + g_e(z), \tag{4.5}$$

where $g_i(z)$ is analytic in the interior of Γ, while $g_e(z)$ is analytic in the exterior of Γ and vanishes at infinity. Inserting the description of Γ given by (4.4) and (4.5) into (4.3) we find that

$$h(z) = g_e(z). \tag{4.6}$$

We can actually determine $g_e(z)$ without having to evaluate all the moment constants to give $h(z)$ via (4.2). If we assume that the initial curve Γ_0 may be expressed as

$$\bar{z} = g^0(z) = g_i^0(z) + g_e^0(z), \tag{4.7}$$

where the decomposition on the right parallels that in (4.5), it follows from (3.3), (4.2) and (4.6) that, when the area has been increased by A, we shall have

$$g_e(z) = g_e^0(z) + A/\pi z. \tag{4.8}$$

We may thus regard $g_e(z)$ as known.

In order to have a complete description of the curve Γ, we need to determine $g_i(z)$, or $f(\zeta)$, from the known function $g_e(z)$. To do this we recognize that not every analytic function $g(z)$ describes a curve through the prescription (4.4). One can easily see that we require $\bar{g}(z) = g^{-1}(z)$: the conjugate function and the inverse function must be identical. This functional relation alone now serves to fix $g_i(z)$. In fact, it proves simplest to determine $f(\zeta)$ instead. For this purpose we observe that, on the unit circle in the ζ plane, we have

$$g\{f(\zeta)\} = \overline{f(\zeta)} = \overline{f(1/\bar{\zeta})}.$$

But the equality between the outer two expressions is an equality between the boundary values of analytic functions and is therefore valid throughout any domain into which the functions may be analytically continued. The decomposition (4.5) allows this functional relation to be written as

$$g_i\{f(\zeta)\} + g_e\{f(\zeta)\} = \overline{f(1/\bar{\zeta})}. \tag{4.9}$$

Knowing $g_e(z)$, this is to furnish $f(\zeta)$. Since $f(\zeta)$ gives a 1–1 map of $|\zeta| \leqslant 1$ onto D, and is analytic within the unit circle, it follows that

(i) the first term on the left is analytic in $|\zeta| \leqslant 1$;

(ii) the second term on the left has singularities in $|\zeta| < 1$ whose nature is the same as that of the known singularities of $g_e(z)$ in D;

(iii) the term on the right is analytic in $|\zeta| \geqslant 1$ and vanishes at infinity.

It follows that the only singularities of $\overline{f(1/\bar{\zeta})}$ are within $|\zeta| < 1$ and that their form is identical with those of $g_e(z)$. One can therefore, in general, write down the *form* of the mapping $f(\zeta)$. When these singularities are merely poles, a quantitative comparison of the singularities in (4.9) will serve to fix $f(\zeta)$ completely by giving algebraic relations determining the positions and principal parts of these poles. The procedure is illustrated by an example in the next section.

5. An example

We consider an initial domain bounded by a circle of radius r, the injection point being off-centre. We may choose axes so that, with injection at the origin, the centre of the initial circle is at the point $z = a$, where a is real and positive and $r > a$. This initial curve is

$$\bar{z} = a + r^2/(z-a) = g^0(z), \tag{5.1}$$

so that

$$g_e^0(z) = r^2/(z-a). \tag{5.2}$$

From (4.8), it follows that, when the area has been increased by A, we have

$$g_e(z) = \frac{r^2}{z-a} + \frac{A}{\pi z}. \tag{5.3}$$

From the functional relation (4.9) it now follows that $\overline{f(1/\bar{\zeta})}$ has two simple poles, one of which is at the origin since $f(0) = 0$. Since it also has to vanish at infinity, it must have the form

$$\overline{f\left(\frac{1}{\bar{\zeta}}\right)} = \frac{\beta}{\zeta-\gamma} + \frac{\delta}{\zeta}, \tag{5.4}$$

whence

$$f(\zeta) = \bar{\beta}\zeta/(1-\bar{\gamma}\zeta) + \bar{\delta}\zeta. \tag{5.5}$$

The principal part of $\overline{f(1/\bar{\zeta})}$ at the origin is δ/ζ.

The principal part of $g_e\{f(\zeta)\}$ at the origin is

$$\frac{A}{\pi}\frac{1}{\bar{\beta}+\bar{\delta}}\frac{1}{\zeta}.$$

Since these must balance in (4.9) we have

$$\delta(\bar{\beta}+\bar{\delta}) = A/\pi.$$

$\overline{f(1/\bar{\zeta})}$ has a simple pole at $\zeta = \gamma$ with residue β. Ensuring that the simple pole of $g_e\{f(\zeta)\}$ is also at $\zeta = \gamma$ gives a second relation, while ensuring that the residue there is β gives a third. In fact, thus far the orientation of the circle in the ζ plane has not been specified, so this system of equations does not uniquely determine β, γ and δ. From the symmetry in this problem we can evidently choose β, γ and δ to be real, with $\delta \geqslant 0$. The mapping then becomes

$$f(\zeta) = \beta\zeta/(1-\gamma\zeta) + \delta\zeta, \tag{5.6}$$

with β, γ and δ to be determined in terms of A, a and r by the system

$$\left.\begin{aligned} \delta(\beta+\delta) &= A/\pi, \\ (a-\gamma\delta)(1-\gamma^2) &= \beta\gamma, \\ (r^2-\beta\delta)(1-\gamma^2)^2 &= \beta^2. \end{aligned}\right\} \tag{5.7}$$

Solving these algebraic equations, extracting that solution for which β, γ and δ vary continuously with A from their initial values at $A = 0$, gives a domain which qualitatively behaves as one would expect; the blob expands, tending towards a circular shape centred on the injection point.

From the above, by letting $r \to \infty$ and $a \to \infty$ while maintaining $r-a = d$ we obtain the result for an initial unbounded blob occupying the half-plane to the right of the line $x = -d$. In this limit $\gamma \to 1$ and we obtain

$$f(\zeta) = \beta\zeta/(1-\zeta) + \delta\zeta, \tag{5.8}$$

where

$$\delta = -\tfrac{1}{3}d + (\tfrac{1}{9}d^2 + A/3\pi)^{\frac{1}{2}}, \quad \beta = 2d + 2\delta. \tag{5.9}$$

As A increases, a bulge moves to the left, while for large $|y|$ the free boundary remains asymptotic to $x = -d$, as one would expect. If we contemplate sucking fluid out, so that A becomes negative, we see that the above solution can only hold down to $A = -\frac{1}{3}\pi d^2$. At this stage the free boundary has developed a cusp, and this singularity will persist when more fluid is removed. A solution scheme designed to follow this physical situation further must allow for this singularity while the present scheme does not. (Physically, the existence of surface tension will prevent a cusp from appearing, of course. Before this stage is reached the effects discussed earlier will render invalid the constant-pressure assumption here employed.) This contrasts with a point mentioned earlier: while *injection* of fluid seems to improve the smoothness properties of the bounding curves, *suction* can produce singularities. We shall not pursue this point, or this particular example, here.

6. An alternative procedure

When the function $g_e(z)$ has branch points, the solution of the functional relation (4.9) is not so easily effected, and the scheme of §4 may be impractical or impossible. In any case, it is only applicable when the initial domain can be described in analytic form. An alternative procedure is possible which in certain circumstances again allows the problem to be reduced exactly to the solution of a finite system of algebraic equations but which, in general, can be used with an associated approximation scheme to effect a similar reduction.

Since $f(\zeta)$ is analytic in $|\zeta| < 1$, and vanishes at the origin, we may there represent it by the series

$$f(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n. \tag{6.1}$$

From (3.2) we find that

$$C_N = \pi \Sigma m a_m a_n \ldots a_r \bar{a}_{m+n+\ldots+r}, \tag{6.2}$$

where there are $N+1$ indices, $m, n, \ldots, r$, and the summation takes each of these indices from 1 to ∞.

If, now, the mapping (6.1) happens to be a finite sum, say a polynomial of degree m so that $a_n = 0$ for $n > m$, it then follows that $C_n = 0$ for $n \geqslant m$. Moreover, the infinite sums of (6.2) become finite. Hence, if the initial domain is such that it is mapped to the unit circle by a polynomial, we may readily determine the initial moment constants from (6.2). Except for C_0, which increases by a known amount, the remainder are invariant, so the mapping at a later time will again be a polynomial of the same order as initially. The coefficients are determined by the finite system obtained by taking those relations of the form (6.2) which are non-vacuous, using the up-dated value of C_0. (As before, some condition must be imposed to fix the orientation of the circle in the ζ plane: for example, we may choose a_1 to be real and positive.)

Generally, the initial mapping will not be exactly a polynomial, but we can approximate a general mapping as such and adopt the above procedure again. In this case, given the initial domain, one can determine the C_N from (3.2), by numerical integration if necessary, followed by a determination of the coefficients from the system obtained in (6.2). Using this routine, one can check the error introduced by employing a polynomial of the degree chosen by comparing the approximate initial domain these coefficients produce when introduced into (6.1) with the initial domain being approximated.

7. Concluding remarks

As was noted in the introduction, the problems analysed in this paper arose from a desire to understand some of the phenomena associated with the more complex injection moulding process. The fact that the motion in the present case is controlled by a simple system of invariants is encouraging, and suggests that a similar approach to problems including interactions with side walls in the mould may be fruitful.

Although the situation discussed here is not the one of primary interest, the invariance of the moment constants, allowing the problem to be posed in a simple mathematical form, provokes a number of questions of a more academic nature. Some of those connected with the moment problem as such are raised at the end of §3, but there are others more closely connected with the physical problem. One can easily envisage an initial simply connected domain having a spiral character which would be such that injection eventually brings two free boundaries together, thus producing a multiply connected region. (At which point, a different solution scheme is needed, of course.) One might ask for the conditions to be imposed on an initial domain to ensure that it remains simply connected. One feels that a condition of star-likeness with respect to the injection point would be sufficient, but certainly not necessary.

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